

A Multiplier Version of the Bernstein Inequality on the Complex Sphere

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Abstract

We prove a multiplier version of the Bernstein inequality on the complex sphere. Included in this is a new result relating a bivariate sum involving Jacobi polynomials and Gegenbauer polynomials, which relates the sum of reproducing kernels on spaces of polynomials irreducibly invariant under the unitary group, with the reproducing kernel of the sum of these spaces, which is irreducibly invariant under the action of the orthogonal group.

Keywords:

Bernstein Inequality, Complex Sphere, Multiplier, Jacobi Polynomial

1. Introduction and preliminaries

In this article we prove a multiplier version of the Bernstein inequality of the type proved by Ditzian [4]. Since the restriction to a geodesic of a polynomial on a complex sphere is just a trigonometric polynomial on a circle, we immediately have a *tangential Bernstein inequality*

$$\|D_u p_n\|_\infty \leq n \|p_n\|_\infty,$$

where D_u is the tangential derivative in the direction of u and p_n is any polynomial of degree n . For more information on tangential Bernstein inequalities on algebraic manifolds see e.g. Bos et. al. [3]. An important stepping stone for this proof is Theorem 2.1, in which prove a new bivariate summation formula for Jacobi polynomials.

We follow Koornwinder [5] in our description of the complex sphere, and the harmonic analysis thereof. Let \mathbb{C}^q be q -dimensional complex space. We will denote vectors in \mathbb{C}^q by $\mathbf{z} = (z_1, z_2, \dots, z_q)$. Let the inner product of two vectors $\mathbf{w}, \mathbf{z} \in \mathbb{C}^q$ be

$$\langle \mathbf{w}, \mathbf{z} \rangle = \sum_{j=1}^q w_j \bar{z}_j,$$

and the length of a vector be $|\mathbf{z}| = \langle \mathbf{z}, \mathbf{z} \rangle^{1/2}$. Let

$$\mathbb{S}^{2q} = \{\mathbf{z} \in \mathbb{C}^q : |\mathbf{z}| = 1\},$$

be the sphere in \mathbb{C}^q . We note here that \mathbb{S}^{2q} has topological dimension $2q - 1$, but that we keep with the established notation so as not to confuse the reader. Let $d(\mathbf{w}, \mathbf{z})$ be the geodesic distance between \mathbf{w} and \mathbf{z} on \mathbb{S}^{2q} .

The complex sphere is invariant under the action of the unitary group \mathcal{U}_q , the group of $q \times q$ complex matrices U which satisfy

$$UU^* = I_q,$$

where $U_{ij}^* = \overline{U_{ji}}$, $i, j = 1, \dots, q$.

Using the polar form for a complex number we can write $\mathbf{z} \in \mathbb{S}^{2q}$ in the form

$$\mathbf{z} = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, \dots, r_q e^{i\phi_q}),$$

where $\sum_{j=1}^q r_j^2 = 1$. If we set $r_1 = \cos \theta$, we can write

$$\mathbf{z} = \cos \theta e^{i\phi} \mathbf{e}_1 + \sin \theta \mathbf{z}', \quad (1)$$

where \mathbf{e}_k is the unit vector in the k th coordinate, and $\mathbf{z}' \in \mathbb{S}^{2(q-1)}$. Here $\phi = \phi_1$, (obviously) $\sin \theta = \sqrt{r_2^2 + \dots + r_q^2}$, and

$$\mathbf{z}' = (\sin \theta)^{-1} (r_2 e^{i\phi_2}, \dots, r_q e^{i\phi_q}).$$

We can easily verify that $\mathbb{S}^{2q} = \{U\mathbf{e}_1, U \in \mathcal{U}_q\}$. Thus, for any $\mathbf{z} \in \mathbb{S}^{2q}$, there exists a $U \in \mathcal{U}_q$ such that $U\mathbf{e}_1 = \mathbf{z}$. We call this action of \mathcal{U}_q on \mathbb{S}^{2q} *transitive*. Now it is clear that if we view \mathcal{U}_{q-2} as acting on the orthogonal complement of \mathbf{e}_1 , then \mathbf{e}_1 remains fixed under this action. Thus we can write

$$\mathbb{S}^{2q} = \frac{\mathcal{U}_q}{\mathcal{U}_{q-1}}.$$

On the real sphere we are accustomed to the idea that the polynomials on the sphere may be orthogonally decomposed into subspaces of spherical harmonics, each of which is invariant under the action of the orthogonal group. For the complex sphere the picture is not so straightforward. Now we wish to identify the spaces of polynomials which are minimally invariant under the action of the unitary group, and this issue is discussed in Section 2.

Let $d\mu_{2q}$ be the \mathcal{U}_q -invariant normalised measure on the sphere, and define the inner product of f, g , two functions on \mathbb{S}^{2q} , by

$$\langle\langle f, g \rangle\rangle = \int_{\mathbb{S}^{2q}} f \bar{g} d\mu_{2q}.$$

Let us define the family of L_r norms on \mathbb{S}^{2q} :

$$\|f\|_r = \begin{cases} \left(\int_{\mathbb{S}^{2q}} |f|^r d\mu_{2q} \right)^{1/r}, & 1 < r < \infty, \\ \text{ess sup} |f|, & r = \infty. \end{cases}$$

In this paper we will be discussing \mathcal{U}_q invariant kernels on \mathbb{S}^{2q} . These are kernels $\kappa : \mathbb{S}^{2q} \times \mathbb{S}^{2q} \rightarrow \mathbb{C}$, such that $\kappa(U\mathbf{x}, U\mathbf{y}) = \kappa(\mathbf{x}, \mathbf{y})$ for all $U \in \mathcal{U}_q$. Previous results of Ditzian [4] have been valid for two-point homogeneous spaces. These are spaces which for pairs of points which are equidistant, there is a single isometry which maps one pair to the other (see Wang [9] for more information). For two points spaces, the geodesic distance is a function of the inner product in the ambient space. A consequence of this is that all isometrically invariant kernels are univariate functions of distance.

For the complex spheres this is not the case. However, we does have the following analogous property. Suppose we have pairs of points $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$, with $\langle \mathbf{x}_1, \mathbf{y}_1 \rangle = \langle \mathbf{x}_2, \mathbf{y}_2 \rangle$. Since the unitary group acts transitively on the complex sphere, there exist $U_1, U_2 \in \mathcal{U}_q$ such that $U_1 \mathbf{x}_1 = U_2 \mathbf{x}_2 = \mathbf{e}_1$. Recalling (1), and using the fact that \mathcal{U}_{q-1} acts transitively on \mathbb{S}^{2q-2} , we know there exists $U' \in \mathcal{U}_q$, such that $U' U_1 \mathbf{y}_1 = U_2 \mathbf{y}_2$, and $U' \mathbf{e}_1 = \mathbf{e}_1$. Hence, $U_2^{-1} U' U_1 \mathbf{x}_1 = \mathbf{x}_2$, and $U_2^{-1} U' U_1 \mathbf{y}_1 = \mathbf{y}_2$. Hence, we conclude that if $\langle \mathbf{x}_1, \mathbf{y}_1 \rangle = \langle \mathbf{x}_2, \mathbf{y}_2 \rangle$ there exists $U \in \mathcal{U}_q$ such that $U \mathbf{x}_1 = \mathbf{x}_2$ and $U \mathbf{y}_1 = \mathbf{y}_2$. This is analogous to the two point homogeneous property of the reals spheres. A straightforward consequence of this is that if κ is \mathcal{U}_q invariant $\kappa(\mathbf{x}_1, \mathbf{x}_2) = \kappa(U \mathbf{x}_1, U \mathbf{x}_2) = \kappa(\mathbf{y}_1, \mathbf{y}_2)$, so that κ is invariant on points with $\langle \mathbf{x}_1, \mathbf{y}_1 \rangle = \langle \mathbf{x}_2, \mathbf{y}_2 \rangle$. Thus we have

Lemma 1.1. *If κ is a \mathcal{U}_q -invariant kernel then*

$$\kappa(\mathbf{x}, \mathbf{y}) = \psi(\langle \mathbf{x}, \mathbf{y} \rangle)$$

for some univariate function ψ .

We can define a convolution of an arbitrary $f \in L_1(\mathbb{S}^{2q})$ function, with a \mathcal{U}_q -invariant kernel $h \in L_1(\mathbb{S}^{2q})$ function:

$$f * \kappa(\mathbf{x}) = \int_{\mathbb{S}^{2q}} f(\mathbf{y}) \psi(\langle \mathbf{x}, \mathbf{y} \rangle) d\mu(\mathbf{y}).$$

It is observed in [5] that we may view \mathbb{C}^q with typical point

$$\mathbf{z} = (x_1 + iy_1, x_2 + iy_2, \dots, x_q + iy_q)$$

as a $2q$ -dimensional real space with variables

$$\mathbf{w} = (x_1, y_1, x_2, y_2, \dots, x_q, y_q).$$

The inner product of two vectors \mathbf{w} and \mathbf{w}' in this space is

$$(\mathbf{w}, \mathbf{w}') = \sum_{j=1}^q (x_j x'_j + y_j y'_j) = \Re \langle \mathbf{z}, \mathbf{z}' \rangle,$$

with $\mathbf{z}' = (x'_1 + iy'_1, x'_2 + iy'_2, \dots, x'_q + iy'_q)$. Hence, a point with standard representation (1) on the complex sphere, has geodesic distance $\cos^{-1}(\cos \theta \cos \phi)$ from the north pole on the associated real sphere.

2. Harmonic analysis

To start with it might be informative to briefly discuss harmonic analysis on the circle as a subset of the complex numbers as opposed to a subset of \mathbb{R}^2 . In the former case we complex Fourier series with a basis $\{1, z^k, \bar{z}^k\}$, $k = 1, \dots$. The unitary group in this case is just the unit circle in the complex numbers. Invariant subspaces under the action of the unitary group are just the one dimensional spaces, constants, $\text{span}\{z^k\}$, $\text{span}\{\bar{z}^k\}$, $k = 1, 2, \dots$. For the latter case we have a basis $\{1, \Re(z^k), \Im(z^k)\}$, $k = 1, 2, \dots$. The subspaces which are invariant under 2×2 orthogonal matrices are, constants, $\text{span}\{\Re(z^k), \Im(z^k)\}$, $k = 1, 2, \dots$, which are two dimensional. Hence we see that the use of the unitary matrices, as opposed to the orthogonal matrices has given us a finer division of the polynomial spaces.

In this spirit let us define the space $\mathcal{P}(m, n)$ of homogeneous polynomials in \mathbb{C}^q as those of the form $P(\mathbf{z}, \bar{\mathbf{z}}) = P(z_1, z_2, \dots, z_q, \bar{z}_1, \bar{z}_2, \dots, \bar{z}_q)$, satisfying

$$P(\alpha \mathbf{z}, \bar{\beta} \bar{\mathbf{z}}) = \alpha^m \bar{\beta}^n P(\mathbf{z}, \bar{\mathbf{z}}), \quad m, n \in \mathbb{N}, \quad \alpha, \beta \in \mathbb{C}.$$

Here we are regarding \mathbf{z} and $\bar{\mathbf{z}}$ formally as different variables, though this is not really the case. Then we define $\text{hom}(m, n)$, the space of homogeneous polynomials on the sphere, to be the restriction of $\mathcal{P}(n, m)$ to the sphere via

$$p(\mathbf{z}) = P(\mathbf{z}, \bar{\mathbf{z}}), \quad \mathbf{z} \in \mathbb{S}^{2q}.$$

Since on the sphere $z_1 \bar{z}_1 + z_2 \bar{z}_2 + \dots + z_q \bar{z}_q = 1$, we have $\text{hom}(m-1, n-1) \subset \text{hom}(m, n)$. We define the space of harmonic polynomials $\text{harm}(m, n) = \text{hom}(m, n) \cap \text{hom}(m-1, n-1)^\perp$, where orthogonality is with respect to the inner product above. In [5] these polynomials are defined in terms of the Laplace operator, but Theorem 3.4 therein:

$$\text{hom}(m, n) = \oplus_{k=0}^{\min(m, n)} \text{harm}(m-k, n-k),$$

tells us that our definition is equivalent. For ease of notation let us write $m \wedge n = \min(m, n)$.

From [5] we know that the dimension of $\text{harm}(m, n)$ is

$$d_{m,n} = \frac{(m+n+q-1)(m+q-2)!(n+q-2)!}{m!n!(q-1)!(q-2)!}. \quad (2)$$

Now, let $\mathcal{H}_l = \oplus_{k=0}^l \text{harm}(l-k, k)$ be the harmonic space of degree l . We can compute the dimension d_l of \mathcal{H}_l directly by summation, but also we have that it has the same dimension as the space of spherical harmonics in \mathbb{R}^{2q} , which from e.g. Müller [6, Page 4] is

$$d_l = \binom{l+2q-1}{l} - \binom{l+2q-3}{l} = \frac{2}{(2q-2)!} \frac{(l+q-1)(l+2q-3)!}{l!}. \quad (3)$$

The dimension of the full polynomial space $\mathcal{P}_n = \oplus_{l=0}^n \mathcal{H}_l$ is

$$t_n = \frac{1}{(2q-1)!} \frac{(2n+2q-1)(n+2q-2)!}{n!}. \quad (4)$$

Let $k_1, k_2, \dots, k_{d_{m,n}}$ be an orthonormal basis for $\text{hom}(m, n)$. Then the reproducing kernel for projection onto $\text{hom}(m, n)$ is

$$\kappa_{m,n}(\mathbf{z}, \mathbf{w}) = \sum_{j=1}^{d_{m,n}} k_j(\mathbf{z}) \bar{k}_j(\mathbf{w}). \quad (5)$$

It is straightforward to show that this kernel is \mathcal{U}_q -invariant. Similarly the reproducing kernel for \mathcal{H}_l ,

$$h_l(\mathbf{z}, \mathbf{w}) = \sum_{k=0}^l \kappa_{l-k,k}(\mathbf{z}, \mathbf{w}),$$

is \mathcal{U}_q -invariant.

Since the reproducing kernels are \mathcal{U}_q -invariant we have, from Lemma 1.1, that

$$\kappa_{m,n}(\mathbf{z}, \mathbf{w}) = \psi(\langle \mathbf{z}, \mathbf{w} \rangle),$$

for some univariate function ψ . In order to determine ψ we need to use its orthogonality properties.

In terms of the standard representation (1) we can write the surface element on \mathbb{S}^{2q} as

$$d\mu_{2q}(\mathbf{z}) = \cos \theta (\sin \theta)^{2q-3} d\theta d\phi d\mu_{2q-2},$$

since surface area on $r\mathbb{S}^{2q-2}$ scales like r^{2q-3} . If we make the change of variable $t = \cos \theta$ then we see that the measure $\cos \theta (\sin \theta)^{2q-3} d\theta = t(1-t^2)^{q-2} dt$ arises.

Thus we might expect the reproducing kernels for the harmonic subspaces, which are orthogonal, to be related to orthogonal polynomials with a weight $(1-t^2)^{q-2}$, and indeed this is the case.

From [5] we have the following representation of the reproducing kernels for the irreducible polynomial spaces $\mathcal{H}(m, n)$,

$$k_{m,n}(\mathbf{x}, \mathbf{y}) = d_{m,n} e^{i(m-n)\phi} (\cos \theta)^{|m-n|} \frac{P_{m \wedge n}^{(q-2, |m-n|)}(\cos(2\theta))}{P_{m \wedge n}^{(q-2, |m-n|)}(1)}, \quad (6)$$

where $\langle \mathbf{x}, \mathbf{y} \rangle = \cos \theta e^{i\phi}$. Here $P_j^{(\alpha, \beta)}$ is the degree j Jacobi polynomial which is orthogonal with respect to the weight $(1-t)^\alpha (1+t)^\beta$. For ease of notation we will now write $\kappa_l(\theta, \phi)$ instead of $\kappa_l(\mathbf{x}, \mathbf{y})$.

As stated in the introduction, we can also view the complex sphere as a real sphere. The harmonics in $\text{hom}(m, n)$ are complex harmonics on the real sphere of degree $m+n$; see [5]. The associated real sphere is of dimension $2q-1$. Hence, we have the following reproducing kernel formula

$$\sum_{m+n=l} \kappa_{m,n}(\mathbf{z}, \mathbf{w}) = h_l(\mathbf{z}, \mathbf{w}) = d_l \frac{P_l^{(q-1)}(\langle \mathbf{z}, \mathbf{w} \rangle)}{P_l^{(q-1)}(1)}, \quad (7)$$

where $P_l^{(\sigma)}, l \geq 0$ are the Gegenbauer polynomials which are orthogonal with respect to the weight $(1-t^2)^{\sigma-1/2}$. We normalise the Gegenbauer polynomials by

$$P_l^{(\sigma)}(1) = \binom{l+2\sigma-1}{l}.$$

Here we interpret \mathbf{z} and \mathbf{w} as points on the real sphere, and if $\langle \mathbf{z}, \mathbf{w} \rangle = e^{i\phi} \cos \theta$ then $(\mathbf{z}, \mathbf{w}) = \cos \theta \cos \phi$ (see the closing remarks of Section 1).

In mind of (5) and (6), we have the following interesting (and we believe new) formula relating Jacobi and Gegenbauer polynomials.

Theorem 2.1. *For $d \geq 1$ and $l \geq 0$*

$$\sum_{m+n=l} d_{m,n} e^{i(m-n)\phi} (\cos \theta)^{|m-n|} \frac{P_{m \wedge n}^{(q-2, |m-n|)}(\cos(2\theta))}{P_{m \wedge n}^{(q-2, |m-n|)}(1)} = d_l \frac{P_l^{(q-1)}(\cos \theta \cos \phi)}{P_l^{(q-1)}(1)}.$$

We wish to define multiplier (pseudodifferential) operators via their action on the harmonic subspaces \mathcal{H}_l . Let M_l be the orthogonal projector from $L_2(\mathbb{S}^{2q}) \rightarrow \mathcal{H}_l$, $l = 0, 1, \dots$. The kernel of this projection is h_l , so that

$$M_l f = f * h_l.$$

Let $\lambda_l, l = 0, 1, \dots$, be a sequence of increasing real numbers. Then, for $f \in L_1(\mathbb{S}^{2q})$ (which thus has a formal Fourier expansion), the multiplier operator

$$\Lambda f = \sum_{l=0}^{\infty} \lambda_l M_l f.$$

In Theorem 4.1, in Section 4, we will show that for $p \in \mathcal{P}_n$,

$$\|\Lambda p\|_r \leq \lambda_n \|p\|_r, \quad 1 \leq r \leq \infty.$$

3. Cesaro means for reproducing kernels

In order to prove Theorem 4.1 we observe that for $p \in \mathcal{P}_n$,

$$\Lambda p = K_m * p, \quad m \geq n,$$

where

$$K_m = \sum_{l=0}^n \lambda_l h_l + \sum_{l=n+1}^m \tilde{\lambda}_l h_l, \quad (8)$$

where the numbers $\tilde{\lambda}_l, l = n+1, \dots, m$ are available for us to choose. Let us define the sequence $\rho_l = \lambda_l, l = 0, 1, \dots, n$ and $\rho_l = \tilde{\lambda}_l, l = n+1, \dots, m$.

Using Young's inequality

$$\|f * g\|_p = \|f\|_p \|g\|_1,$$

we are directed towards the computation of the 1-norms of the kernels K_m , which we achieve via the Cesaro means of h_l

$$S_m^\delta = \frac{1}{C_m^\delta} \sum_{l=0}^m C_{m-l}^\delta h_l,$$

where

$$C_k^\delta = \binom{k+\delta}{k} \asymp k^\delta, \quad k = 0, 1, \dots, m.$$

Before we proceed we need a preliminary technical lemma:

Lemma 3.1. *Let (see [8, 4.7.15])*

$$\begin{aligned} \gamma_l^{(\sigma)} &= \int_{-1}^1 (1-t)^{\sigma-1/2} |P_l^{(\sigma)}(t)|^2 dt \\ &= \frac{2^{1-2\sigma} \pi \Gamma(l+2\sigma)}{(\Gamma(\sigma))^2 (l+\sigma) l!}. \end{aligned}$$

Then, for $l > 0$,

$$d_l = \dim \mathcal{H}_l = \frac{2^{(3-2q)} \pi (2q-3)! (P_l^{(q-1)}(1))^2}{(q-1)!(q-2)! \gamma_l^{(q-1)}}.$$

Proof: Starting from (3), a straightforward calculation gives us

$$\begin{aligned}
d_l &= \frac{2}{(2q-2)!} \frac{(l+q-1)(l+2q-3)!}{l!} \\
&= \frac{2}{(2q-2)!} \frac{2^{(3-2q)}\pi((2q-3)!)^2}{((q-2)!)^2} \left\{ \frac{((q-2)!)^2}{2^{(3-2q)}\pi} \frac{(l+q-1)l!}{(l+2q-3)!} \right\} \left(\frac{(l+2q-3)!}{l!(2q-3)!} \right)^2 \\
&= \frac{2^{(3-2q)}\pi(2q-3)!}{(q-1)!(q-2)!} \frac{(P_l^{(q-1)}(1))^2}{h_l^{(q-1)}}. \quad \blacksquare
\end{aligned}$$

Using this last result and (7) we see that

$$S_n^\delta(\cos \psi) = \frac{2^{(3-2q)}\pi(2q-3)!}{(q-1)!(q-2)!} \frac{1}{C_n^\delta} \sum_{l=0}^n C_{l-n}^\delta \frac{P_l^{(q-1)}(1)P_l^{(q-1)}(\cos \psi)}{h_l^{(q-1)}},$$

where $\cos \psi = \cos \theta \cos \phi$, in other words are essentially the Cesaro means of the Gegenbauer polynomials.

Using Equation [8, 4.5.3] and Lemma 3.1 we have the following corollary of Theorem 2.1:

Corollary 3.2. *The reproducing kernel for \mathcal{P}_n is*

$$r_n((\mathbf{z}, \mathbf{w})) = \sum_{l=0}^n h_l = t_n \frac{P_n^{(q+1/2, q-1/2)}(\cos \theta \cos \phi)}{P_n^{(q+1/2, q-1/2)}(1)},$$

with $(\mathbf{z}, \mathbf{w}) = \cos \theta \cos \phi$, where we recall that $t_n = \dim(\mathcal{P}_n)$.

To estimate these we use the the following results which are given in Bonami and Clerk [2, Page 230].

Proposition 3.3. *If $0 \leq \delta \leq q$ then there is a constant C such that,*

$$S_n^\delta(\cos \psi) \leq C \begin{cases} n^{q-\delta-1}\psi^{-(q+\delta)}, & 3/n \leq \psi \leq \pi/4, \\ n^{2q-1}, & 0 \leq \psi \leq 3/n. \end{cases} \quad (9)$$

In the remainder of this paper the number C will be used to denote a constant which is independent of n .

The main result of this section is

Theorem 3.4. *For $0 \leq \delta \leq q$,*

$$\|S_n^\delta\|_1 \leq C \begin{cases} n^{q-1-\delta}, & \delta \leq q-2, \\ (\log n)^2, & \delta = q-1, \\ 1, & \delta \geq q. \end{cases}$$

Proof: We will provide a bound for

$$\begin{aligned}
\|S_n^\delta\|_1 &= \int_{\mathbb{S}^{2q}} |S_n^\delta((\mathbf{z}, \mathbf{e}))| d\mu(\mathbf{z}) \\
&= \int_0^{2\pi} \int_0^{\pi/2} \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi \\
&= 2 \int_0^\pi \int_0^{\pi/2} \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi.
\end{aligned}$$

Suppose that Q is the region $\pi/4 \leq \theta \leq \pi/2$ or $\pi/4 \leq \phi \leq \pi$. Then $\cos \theta \cos \phi \leq 1/\sqrt{2}$. Setting $\cos \psi = \cos \theta \cos \phi$, we have $\pi/4 \leq \psi \leq 3\pi/4$. Since, from Proposition 3.3, $S_\nu^\delta(\cos \psi)$ is bounded above for $\pi/4 \leq \psi \leq 3\pi/4$ we have

$$I_1 = \int_Q \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi \leq C. \quad (10)$$

We break the remaining integral into 4 parts, $(\theta, \phi) \in [0, 1/n]^2$, $[0, 1/n] \times [1/n, \pi/4]$, $[1/n, \pi/4] \times [0, 1/n]$ and $[1/n, \pi/4] \times [1/n, \pi/4]$ which we call I_2, I_3, I_4 and I_5 respectively. Firstly, since on $[0, 1/n]^2$, $\cos \theta \cos \phi \geq \cos^2(1/n) \geq \cos(3/n)$ (this is easy to check), we have

$$\begin{aligned}
I_2 &= \int_0^{1/n} \int_0^{1/n} \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi \\
&\leq C n^{2q-1} \int_0^{1/n} \left(\int_0^{1/n} \cos \theta (\sin \theta)^{2q-3} d\theta \right) d\phi \\
&\leq C.
\end{aligned} \quad (11)$$

Now, for the remaining integrals we observe that

$$\begin{aligned}
\psi &= \arccos(\cos \theta \cos \phi) \\
&\leq C(1 - \cos \theta \cos \phi)^{1/2},
\end{aligned}$$

since $\arccos(z) \leq C(1 - z)^{1/2}$ as $z \rightarrow 1$. Now, for $0 \leq \theta, \phi \leq 1$,

$$\begin{aligned}
1 - \cos \theta \cos \phi &\geq 1 - \left(1 - \frac{\theta^2}{2}\right) \left(1 - \frac{\phi^2}{2}\right) \\
&= \frac{\theta^2 + \phi^2}{2} - \frac{\theta^2 \phi^2}{4} \\
&\geq \frac{\theta^2 + \phi^2}{4}.
\end{aligned}$$

If we use this last equation in (9), we see that

$$|S_n^\delta(\cos \psi)| \leq C n^{q-\delta-1} (1 - \cos \theta \cos \phi)^{-(q+\delta)/2} \leq C n^{q-\delta-1} (\theta^2 + \phi^2)^{-(q+\delta)/2}. \quad (12)$$

We note that for any $\alpha, \beta > 0$,

$$(\theta^2 + \phi^2)^{-(\alpha+\beta)} \leq \theta^{-2\alpha} \phi^{-2\beta}.$$

We have

$$\begin{aligned} I_3 &= 2 \int_{1/n}^{\pi/4} \int_0^{1/n} \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi \\ &\leq C n^{q-1-\delta} \int_0^{1/n} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi. \end{aligned}$$

Now if $\delta \leq q-2$, then

$$\begin{aligned} \int_{1/n}^{\pi/4} \int_0^{1/n} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi &\leq \int_{1/n}^{\pi/4} \int_0^{1/n} (\theta^2 + \phi^2)^{q-3/2-(q+\delta)/2} d\theta d\phi \\ &\leq \int_{1/n}^{\pi/2} r^{(q-\delta-3)} r dr < C, \end{aligned}$$

using a change to polar coordinates. If $\delta = q-1$, we have (assuming $q > 1$ and using $(\theta^2 + \phi^2)^{-q+1} \leq \theta^{-2q+5/2} \phi^{-1/2}$)

$$\begin{aligned} \int_0^{1/n} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-q+1/2} d\theta d\phi &\leq \int_0^{1/n} \phi^{-1/2} d\phi \int_{1/n}^{\pi/4} \theta^{-1/2} d\theta \\ &\leq C. \end{aligned}$$

If $\delta = q$, we have (using $(\theta^2 + \phi^2)^{-q+1} \leq \theta^{-2q+5/2} \phi^{-1/2}$)

$$\begin{aligned} \int_0^{1/n} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-q} d\theta d\phi &\leq \int_0^{1/n} \phi^{-1/2} d\phi \int_{1/n}^{\pi/4} \theta^{-5/2} d\theta \\ &\leq Cn. \end{aligned}$$

Putting these estimates for the integral into (11) we have

$$I_3 \leq C \begin{cases} n^{q-1-\delta}, & \delta \leq q-2, \\ 1, & \delta \geq q-2. \end{cases} \quad (13)$$

More straightforwardly,

$$\begin{aligned} I_4 &= 2 \int_{1/n}^{\pi/4} \int_0^{1/n} \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi \\ &\leq C n^{q-1-\delta} \int_{1/n}^{\pi/4} \int_0^{1/n} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi \\ &\leq C n^{q-1-\delta} \int_{1/n}^{\pi/4} \phi^{-q-\delta} d\phi \int_0^{1/n} \theta^{2q-3} d\theta \\ &\leq C n^{q-1-\delta+q+\delta-1-2q+2} = C. \end{aligned}$$

For the last integral

$$\begin{aligned} I_5 &= 2 \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \cos \theta (\sin \theta)^{2q-3} |S_n^\delta(\cos \theta \cos \phi)| d\theta d\phi \\ &\leq C n^{q-1-\delta} \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi. \end{aligned} \quad (14)$$

Now if $\delta \leq q-2$, then

$$\begin{aligned} \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi &\leq \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} (\theta^2 + \phi^2)^{q-3/2-(q+\delta)/2} d\theta d\phi \\ &\leq \int_{1/n}^{\pi/2} r^{(q-\delta-3)} r dr < C. \end{aligned}$$

If $\delta = q-1$ we have

$$\begin{aligned} \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi &\leq \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q-1/2)} d\theta d\phi \\ &\leq \int_{1/n}^{\pi/4} \phi^{-1} d\phi \int_{1/n}^{\pi/4} \theta^{-1} d\theta \\ &\leq C(\log n)^2. \end{aligned}$$

For the last case, $\delta = q$, we have

$$\begin{aligned} \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-(q+\delta)/2} d\theta d\phi &\leq \int_{1/n}^{\pi/4} \int_{1/n}^{\pi/4} \theta^{2q-3} (\theta^2 + \phi^2)^{-2q} d\theta d\phi \\ &\leq \int_{1/n}^{\pi/4} \phi^{-3/2} d\phi \int_{1/n}^{\pi/4} \theta^{-3/2} d\theta \\ &\leq Cn. \end{aligned}$$

Hence, substituting the above estimates into (14), we see that

$$I_5 \leq C \begin{cases} n^{q-1-\delta}, & \delta \leq q-2, \\ (\log n)^2, & \delta = q-1, \\ 1, & \delta = q. \end{cases} \quad (15)$$

A simple inspection of the bounds (10) to (15) tells us that the bound for (14) is the largest, giving the required result. Estimates for I_6 and I_7 can be obtained similarly. \blacksquare

4. The Bernstein inequality

The Laplace Beltrami operator on the complex sphere has eigenspaces \mathcal{H}_l , with eigenvalue $\lambda_l = l(l+2q-1)$, $l = 0, 1, \dots$; see e.g. [7]. Thus the fractional order differential operator Λ has multipliers $\lambda_l = (l(l+2q-1))^{\gamma/2}$.

Performing Abel summation $q + 1$ times on (8) we get

$$K_m = \sum_{l=1}^m \Delta^{q+1} \rho_l C_l^q S_l^q + \sum_{l=0}^q \Delta^l \rho_{m-l} C_{m-l}^l S_{m-l}^l, \quad (16)$$

where $\Delta^0 \rho_k = \rho_k$, $\Delta \rho_k = \rho_k - \rho_{k+1}$, and $\Delta^j \rho_k = \Delta(\Delta^{j-1} \rho_k)$, $j = 2, 3, \dots$.

Now let us define

$$g(x) = \begin{cases} 1, & 0 \leq x < n, \\ 1 - C_{n,q} \int_n^x |(y-n)(2n-y)|^{q+1} dy, & n \leq x \leq 2n, \\ 0, & x > 2n, \end{cases}$$

where

$$C_{n,q} = \left(\int_n^{2n} |(y-n)(2n-y)|^{q+1} dy \right)^{-1} = \frac{(2q+3)!}{n^{2q+3}((q+1)!)^2}.$$

This last equations follows by making the change of variable $ns = y - n$ in the above integral, giving

$$\begin{aligned} \int_n^{2n} |(y-n)(2n-y)|^{q+1} dy &= n^{2q+3} \int_0^1 s^{q+1} (1-s)^{q+1} ds \\ &= n^{2q+3} B(q+2, q+2), \end{aligned}$$

where B is the Beta function. We now use $B(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$; (see [1, Page 258]). We have $g \in C^{(q+1)}(\mathbb{R}_+)$, and a simple computation shows that, for $1 \leq j \leq q+1$,

$$g^{(j)}(x) = C_{q,n} \sum_{k=1}^{\lfloor (j+1)/2 \rfloor} \nu_k ((x-n)(2n-x))^{q+1-j+k} (3-2x)^{j+1-2k}, \quad n \leq x \leq 2n,$$

and is zero otherwise. Here, the ν_k depend on q , and $\lfloor \cdot \rfloor$ denotes the integer part. Hence, for $1 \leq j \leq q+1$, we can bound

$$|g^{(j)}(x)| \leq C n^{-j}.$$

Let $h(x) = (x(x+2q-1))^{\gamma/2}$, and $f = gh$, and $\rho_k = f(k)$. We have then, for $0 \leq k \leq n$, $\rho_k = \lambda_k$. We observe that $\rho \in C^{(q+1)}(\mathbb{R}_+)$, where $\mathbb{R}_+ = \{x : x \geq 0\}$. We can estimate the difference

$$|\Delta^l \rho_k| \leq C \max_{x \in [k, k+l]} |f^{(l)}(x)|, \quad l = 0, 1, \dots, q+1.$$

Using Leibnitz rule we have, for $n \leq x \leq 2n+l$,

$$|f^{(l)}(x)| = \left| \sum_{j=0}^l \binom{l}{j} g^{(j)} h^{(l-j)}(x) \right|$$

$$\begin{aligned}
&\leq C \sum_{j=0}^l n^{-j} n^{\gamma-l+j} \\
&\leq C n^{\gamma-l}.
\end{aligned}$$

Therefore, for $0 \leq l \leq q+1$,

$$|\Delta^l \rho_k| \leq C \begin{cases} (k+l)^{\gamma-l}, & 0 \leq k \leq n-l, \\ n^{\gamma-l+j}, & n-l \leq k \leq n+l, \\ 0, & \text{otherwise.} \end{cases}$$

If we substitute these estimates, with Theorem 3.4 into (16) we see that, if $m \geq 2n+q+1$ (and so $|\Delta^l \rho_{m-l}| = 0$, for $l = 0, 1, \dots, q$)

$$\begin{aligned}
\|K_m\|_1 &\leq \sum_{l=1}^m |\Delta^{q+1} \rho_l| C_l^q \|S_l^q\|_1 \\
&\leq C \left(\sum_{l=1}^{n-q-1} l^{\gamma-q-1} l^q + \sum_{l=n-q}^{2n+q+1} n^{\gamma-q-1} l^q \right) \\
&\leq C n^\gamma.
\end{aligned}$$

Thus we have

Theorem 4.1. For $\lambda_l = (l(l+2d-1))^{\gamma/2}$,

$$\|\Lambda p\|_r \leq C n^\gamma \|p\|_r, \quad 1 \leq r \leq \infty,$$

for every $p \in \mathcal{P}_n$.

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